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CALCULATION OF THE LOW NATURAL FREQUENCIES OF CLAMPED CYLINDRICAL SHELLS BY ASYMPTOTIC METHODS

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Abstract—The low natural frequencies of a fully clamped, elastically isotropic circular cylindrical shell are calculated using asymptotic techniques. Two types of expansion are considered. The first assumes a given, low number of circumferential waves. To a first approximation, the frequencies are shown to be given by membrane theory, and a table of these values is obtained numerically for various shell lengths, circumferential wave numbers and a Poisson's ratio of 0·3. The bending correction is shown to be proportional to the square root of the thickness–radius ratio, to vanish with Poisson's ratio, and to be expressible solely in terms of the solutions of the membrane eigenvalue problem. The second type of expansion is necessary to calculate the minimum natural frequency of a shell of given length and allows the number of circumferential waves to approach infinity as the thickness–radius ratio approaches zero. A remarkably simple formula for the frequencies is obtained in terms of the natural frequencies of a clamped–clamped beam. The two asymptotic results are shown to overlap.

INTRODUCTION

EXACT solutions, in terms of tabulated functions, for the natural frequencies and mode shapes of elastic shells exist only for the simplest midsurface geometries and boundary conditions. However, since the governing equations, suitably nondimensionalized, exhibit the flexural rigidity of the shell only through a small parameter β, it seems natural to exploit this fact and to seek approximate asymptotic solutions. In static problems, asymptotic (or perturbation) methods have been employed for over half a century since the work of Reissner [1], but, apparently, it is only with the recent appearance of a series of papers by Gol'denveiser [2, 3], and Ross [4–7] that such methods have been systematically applied to shell vibrations.

In his studies (which have been confined to shells of revolution) Ross has pointed out that certain of his techniques and conclusions have to be modified for cylindrical shells. Because of the technical importance of these shells, further detailed analyses seem warranted. There exists already, of course, a vast body of literature on cylinder vibrations. However, in our opinion, the governing equations employed heretofore have been either overly simplified (Donnell-type approximations) and thus have failed to accurately predict certain important phenomena or unnecessarily complicated (Flügge-type approximations), and thus have buried essential features beneath a mound of algebra. Furthermore, many previous studies, being based on Rayleigh–Ritz or strictly numerical methods, have failed to explicate the order of magnitude relations among the various dimensionless parameters that enter the problem. It has become a truism to remark that such relations are as important to understanding the physics of a problem as specific numerical results.

In the present paper we present two different asymptotic approximations for calculating the *low* natural frequencies of an elastically isotropic, fully clamped circular cylindrical shell. Low frequencies are such that rapid *axial* variations of the mode shapes are confined to narrow zones at the ends of the cylinder.

The analysis starts from Rayleigh's principle. By working with Koiter's modified strain energy density [8, 9], we obtain the simplest possible set of differential equations and boundary conditions in terms of displacement components. When cast into dimensionless form, these equations contain the parameters ν, l, m, λ and β, where ν is Poisson's ratio, Rl is the length of the shell, R is the midsurface radius, m is the circumferential wave number,

$$\lambda = \frac{\rho R^2 \omega^2}{E}, \tag{1}$$

ρ is the mass per unit volume, ω is the frequency, E is Young's modulus,

$$\beta^4 = \frac{h^2}{12(1-\nu^2)R^2} \tag{2}$$

and h is the thickness, taken constant. Low natural frequencies are such that $\lambda < 1$.

In the first type of asymptotic expansion considered, we assume that m is a given, $0(1)$ integer. The dimensionless natural frequencies λ_n, $n = 1, 2, \ldots$, hereafter called the eigenvalues, turn out, to a first approximation, to be identical to those predicted by *membrane theory* and are given by an asymptotic formula of the form

$$\lambda_n(l, \nu, m, \beta) = \Lambda_n(l, \nu, m) + \beta \lambda_n^1(l, \nu, m) + \ldots. \tag{3}$$

A fairly comprehensive table of the membrane eigenvalues Λ_n is presented, Table 1, together with an explicit formula for λ_n^1 in terms of the *solution of the membrane eigenvalue problem*. Physically, it is meaningless to compute terms beyond λ_n^1.

If the lowest eigenvalue of a shell of given length is required, then one must consider circumferential wave numbers that grow arbitrarily large as the thickness-radius ratio approaches zero. A second type of asymptotic expansion is then necessary to account for bending effects which are now significant over the entire shell. However, to a first approximation, the governing equations turn out to be actually simpler than those for the first type of asymptotic expansion and lead to an expression for λ_n in terms of the eigenvalues of a clamped–clamped beam. The two different asymptotic approximations to λ_n are shown to have a common region of validity.

The power of asymptotic methods cannot be overemphasized. For example, in contrast to Forsberg's elaborate, analytical/numerical computations of the lowest eigenvalue of a clamped cylindrical shell [10], our simple equation (88) shows explicitly the dependence of the lowest eigenvalue (as well as nearby higher ones) on m, β and l.

TABLE 1. LOW NATURAL FREQUENCIES FOR CLAMPED MEMBRANES: $\nu = 0{\cdot}3$, $\Lambda_n = \rho R^2\omega_n^2/E$
ALL EIGENVALUES WITH AN ASTERISK (*) CORRESPOND TO ANTISYMMETRIC MODES

n \ l	1	1·5	2	3	4	5
	circumferential wave number $m = 1$					
1	0·79935	0·55703	0·37365	0·18307	0·10248	0·06297
2	0·94425*	0·87546*	0·78253*	0·55658*	0·36084*	0·23623*
3	0·98083	0·95048	0·90707	0·78544	0·63368	0·48434
4	0·98597*	0·96841*	0·94370*	0·87312*	0·77830*	0·67091*
5	0·99300	0·98247	0·96719	0·92312	0·86156	0·71864*
6	0·99378*	0·98593*	0·97482*	0·94138*	0·86473*	0·78434
	circumferential wave number $m = 2$					
1	0·47198	0·26079	0·15443	0·06500	0·03205	0·01746
2	0·81622*	0·63822*	0·46826*	0·24242*	0·13340*	0·07909*
3	0·91702	0·81753	0·69924	0·46819	0·29936	0·19350
4	0·95051*	0·89190*	0·81608*	0·63819*	0·46790*	0·33380*
5	0·96966	0·93058	0·87855	0·74715	0·60215	0·46780
6	0·97772*	0·95048*	0·91354*	0·81593*	0·69889*	0·57851*
	circumferential wave number $m = 3$					
1	0·28074	0·13458	0·07254	0·02636	0·01152	0·00570
2	0·65241*	0·41996*	0·26398*	0·11392*	0·05609*	0·03043*
3	0·82291	0·65242	0·48961	0·26141	0·14360	0·08365
4	0·89456*	0·78095*	0·65234*	0·41895*	0·25960*	0·16324*
5	0·93187	0·85249	0·75571	0·55167	0·38052	0·25856
6	0·95144*	0·89433*	0·82130*	0·65230*	0·48927*	0·35650*
	circumferential wave number $m = 4$					
1	0·17523	0·07559	0·03763	0·01206	0·00483	0·00225
2	0·49726*	0·27210*	0·15423*	0·05870*	0·02645*	0·01338*
3	0·71410	0·49715	0·30116	0·14942	0·07401	0·04002
4	0·82343*	0·65720*	0·49698*	0·26776*	0·14659*	0·08451*
5	0·88240	0·75863	0·62357	0·38907	0·23536	0·14487
6	0·91605*	0·82313*	0·71364*	0·49685*	0·32863*	0·21575*
	circumferential wave number $m = 5$					
1	0·11460	0·04527	0·02104	0·00610	0·00230	0·00103
2	0·37193*	0·17990*	0·09465*	0·03261*	0·01368*	0·00657*
3	0·60419	0·37087	0·22380	0·08930	0·04093	0·02090
4	0·74345*	0·53804*	0·36999*	0·17277*	0·08633*	0·04669*
5	0·82428	0·65940	0·50041	0·27064	0·14768	0·08452
6	0·87311*	0·74324*	0·60419*	0·36918*	0·21934*	0·13287*
	circumferential wave number $m = 6$					
1	0·07791	0·02849	0·01248	0·00335	0·00121	0·00053
2	0·27808*	0·12241*	0·06071*	0·01925*	0·00764*	0·00354*
3	0·50253	0·27552	0·15382	0·05578	0·02404	0·01175
4	0·66063*	0·43323*	0·27379*	0·11410*	0·05315*	0·02736*
5	0·76070	0·56304	0·39526	0·18909	0·09521	0·05155
6	0·82442*	0·66057*	0·50227*	0·27210*	0·14807*	0·08430*

THE GOVERNING EQUATIONS

Let Rx and $R\theta$ denote, respectively, axial and circumferential distance along the midsurface of the circular cylindrical shell. Assuming the midsurface displacement vector to be harmonic in time, of the form $R\mathbf{U}(x, \theta)\, e^{i\omega t}$, we may characterize the eigenvalues λ_n by the well-known condition

$$\lambda_n = \min \int_0^{2\pi} \int_0^l \Phi(\mathbf{U}) \mathrm{d}x\, \mathrm{d}\theta \Big/ \int_0^{2\pi} \int_0^l \mathbf{U} \,.\, \mathbf{U}\, \mathrm{d}x\, \mathrm{d}\theta, \tag{4}$$

where the minimum is taken over all sufficiently smooth, non-zero, displacement fields satisfying the kinematic boundary conditions and mutually orthogonal to the eigenvectors $\mathbf{U}_1, \mathbf{U}_2, \ldots \mathbf{U}_{n-1}$. In (4) $EhR^2\Phi$ is the strain energy density, a homogeneous, positive definite quadratic functional of the extensional and bending strains. These, in turn, depend on $\mathbf{U}$ and its derivatives.

In what follows we shall employ the strain measures of Sanders [11, 12] and Koiter [13]. In terms of U, V and W, the axial, circumferential, and outward radial components of $\mathbf{U}$, these strains read

$$E_x = U', \qquad E_\theta = V^{\cdot} + W, \qquad E_{x\theta} = \tfrac{1}{2}(U^{\cdot} + V') \tag{5}$$

$$K_x = W'', \qquad K_\theta = W^{\cdot\cdot} - V^{\cdot}, \qquad K_{x\theta} = W'^{\cdot} + \tfrac{1}{4}U^{\cdot} - \tfrac{3}{4}V', \tag{6}$$

where primes and dots denote, respectively, differentiation with respect to x and θ. Furthermore, we shall take Φ in the form proposed by Koiter [8, 9]:

$$\Phi = \Phi_M + \Phi_B + \Phi_C, \tag{7}$$

where

$$\Phi_M = \tfrac{1}{2}\eta[E_x^2 + E_\theta^2 + 2\nu E_x E_\theta + 2(1-\nu)E_{x\theta}^2] \tag{8}$$

$$\Phi_B = \tfrac{1}{2}\beta^4[K_x^2 + K_\theta^2 + 2\nu K_x K_\theta + 2(1-\nu)K_{x\theta}^2] \tag{9}$$

$$\Phi_C = \beta^4[E_\theta(K_x + K_\theta) - (1-\nu)E_x K_\theta + (1-\nu)E_{x\theta}K_{x\theta} + \tfrac{1}{2}E_\theta^2 - \tfrac{3}{4}(1-\nu)E_{x\theta}^2] \tag{10}$$

and

$$\eta = \frac{1}{1-\nu^2}. \tag{11}$$

As the subscripts suggest, the three terms composing Φ may be referred to as the membrane, bending, and coupling energy densities. The coupling energy density Φ_C can be shown [8] to make a negligible contribution to Φ, but by its inclusion we can, with the aid of (5) and (6), express Φ in the alternate form

$$\Phi = \Phi_M + \Phi_R + \Phi_D, \tag{12}$$

where

$$\Phi_R = \tfrac{1}{2}\beta^4(W'' + W^{\cdot\cdot} + W)^2 \tag{13}$$

may be referred to as the *reduced* part of the bending energy and

$$\Phi_D = (1-\nu)\beta^4[(-W'W^{\cdot\cdot} - U'V + U^{\cdot}W^{\cdot} + V^{\cdot}W')' + (W'W'' + U'V - U'W - V'W')^{\cdot}] \tag{14}$$

may be referred to as the *divergence* part of the bending energy. Now the surface integral of Φ_D may be replaced by a line integral, which for full clamping (and certain other boundary conditions) vanishes. Hence (4) may be replaced by the simpler expression

$$\lambda_n = \min \int_0^{2\pi} \int_0^l [\Phi_M(U, V, W) + \Phi_R(W)]\, dx\, d\theta \Big/ \int_0^{2\pi} \int_0^l \mathbf{U} \cdot \mathbf{U}\, dx\, d\theta. \tag{15}$$

Because of periodicity and separability, we may restrict attention to displacement fields of the form

$$U = u_m(x) \cos m(\theta + \phi) \tag{16}$$

$$V = v_m(x) \sin m(\theta + \phi) \tag{17}$$

$$W = w_m(x) \cos m(\theta + \phi), \tag{18}$$

where m is a non-negative integer and ϕ is an arbitrary phase angle. When these expressions are substituted into (15), we obtain, upon using (5), (8) and (13), the more explicit condition

$$\lambda_{mn} = \min \int_0^l \tfrac{1}{2}\{\eta[(u_m')^2 + (w_m + mv_m)^2 + 2\nu(u_m')(w_m + mv_m) + \tfrac{1}{2}(1-\nu)(v_m' - mu_m)^2]$$

$$+ \beta^4[w_m'' + (1-m^2)w_m]^2\}\, dx \Big/ \int_0^l \mathbf{u}_m \cdot \mathbf{u}_m\, dx, \tag{19}$$

where the minimum is taken over all sufficiently smooth, non-zero, vector fields

$$\mathbf{u}_m = (u_m, v_m, w_m) \tag{20}$$

satisfying the kinematic conditions of full clamping,

$$u_m = v_m = w_m = w_m' = 0 \quad \text{at } x = 0, l, \tag{21}$$

and mutually orthogonal to the eigenfunctions $\mathbf{u}_{m,1}, \mathbf{u}_{m,2}, \ldots, u_{m,n-1}$. In the following we shall suppress the subscript m without risk of ambiguity.

By the calculus of variations, the solutions of (19) must satisfy the differential system

$$(\beta^4 B + \eta M + \lambda I)\mathbf{u} = 0, \tag{22}$$

subject to the boundary conditions (21). Here, I is the identity matrix,

$$M = [M_{ij}(D, m, \nu)] = \begin{bmatrix} D^2 - \frac{1}{2}m^2(1-\nu) & \frac{1}{2}m(1+\nu)D & \nu D \\ -\frac{1}{2}m(1+\nu)D & \frac{1}{2}(1-\nu)D^2 - m^2 & -m \\ -\nu D & -m & -1 \end{bmatrix}, \tag{23}$$

$D(\ ) = (\ )'$, and the only non-zero element of B is

$$B_{33}(D, m) = -(D^2 + 1 - m^2)^2. \tag{24}$$

ASYMPTOTIC FORMS FOR $m = 0(1)$

The solutions of (22) are linear combinations of the exponentials e^{px}, where the values of p are roots of the bi-quartic polynomial

$$\begin{aligned} -2\det[\beta^4 B(p,m)+\eta M(p,m,\nu)+\lambda I] &= \beta^4(p^2+1-m^2)^2[(1-\nu)\eta^2(p^2-m^2)^2+2\lambda^2 \\ &\quad +(3-\nu)\eta\lambda(p^2-m^2)]+(1-\nu)\eta^2(1-\lambda)p^4 \\ &\quad +\eta\lambda[(1-\nu)\eta(2m^2+3+2\nu)-(3-\nu)\lambda]p^2 \\ &\quad +\lambda[(m^2+1)\eta-\lambda][2\lambda-(1-\nu)\eta m^2]. \end{aligned} \tag{25}$$

If $\lambda \neq 1$, then for β sufficiently small, (25) possesses four regular (or membrane) roots of the form*

$$p = p_M^0+\beta^4 p_M^1+\ldots, \tag{26}$$

where

$$\det[M(p_M^0, m, \nu)+\lambda I] = 0, \tag{27}$$

plus four singular (or bending) roots of the form

$$p = \beta^{-1}p_B^0+p_B^1+\ldots, \tag{28}$$

where

$$(p_B^0)^4+1-\lambda = 0. \tag{29}$$

At the "transition value", $\lambda = 1$, there are only two regular roots but six singular ones and, as Ross notes, the distinction between membrane and bending roots becomes blurred. *In the remainder of the paper we shall confine ourselves to the low frequency case* $\lambda < 1$.

It is convenient to introduce the left and right boundary layer coordinates

$$y = \frac{x}{\beta}, \qquad z = \frac{l-x}{\beta}. \tag{30}$$

The form of (26) and (28) and the fact that the boundary conditions at $x = 0$ and $x = l$ are identical indicates that the solution of (22) has the form

$$\mathbf{u} = \mathbf{u}_M(x,\lambda,\beta)+\beta^q[\mathbf{u}_B(y,\lambda,\beta)+\mathbf{u}_B(z,\lambda,\beta)], \tag{31}$$

where

$$\lim_{y\to\infty} \mathbf{u}_B(y,\lambda,\beta) = \mathbf{0}, \tag{32}$$

and the constant q is to be chosen so that the largest component of $\mathbf{u}_B$ is 0(1). We may assume that the largest component of $\mathbf{u}_M$ is 0(1) without loss of generality because any constant times $\mathbf{u}$ is also a solution.

The vectors $\mathbf{u}_M$ and $\mathbf{u}_B$ both satisfy the differential system (22). A simple order of magnitude analysis reveals that all the components of $\mathbf{u}_M$ are, in general, 0(1) whereas

$$\mathbf{u}_B = (u_B, v_B, w_B) = (\beta f, \beta^2 g, w_B), \tag{33}$$

* To avoid a cluttered notation, we shall not distinguish between superscripts and powers. The proper interpretation will always be clear from the context.

where f and g are O(1). It is convenient to work with a vector all of whose components are O(1) so we set

$$\mathbf{v}_B = (f, g, w_B). \tag{34}$$

Henceforth adopting the convention that D denotes differentiation with respect to the argument of the function on which it operates, we may characterize $\mathbf{u}_M$ as the regular solution of the differential system

$$[\eta M(D, m, \nu) + \beta^4 B(D, m) + \lambda I]\mathbf{u}_M = 0, \tag{35}$$

and $\mathbf{v}_B$ is the decaying solution of the differential system

$$\{B^0(D) + \eta M^0(D, m, \nu) + \beta^2[B^2(D, m) + \eta M^2(D, m, \nu)] + \beta^4 B^4(m) + \lambda I]\mathbf{v}_B = 0, \tag{36}$$

where

$$M^0 = \begin{bmatrix} D^2 & 0 & \nu D \\ -\tfrac{1}{2}m(1+\nu)D & \tfrac{1}{2}(1-\nu)D^2 & -m \\ -\nu D & 0 & -1 \end{bmatrix} \tag{37}$$

$$M^2 = \begin{bmatrix} -\tfrac{1}{2}m^2(1-\nu) & \tfrac{1}{2}m(1+\nu)D & 0 \\ -0 & -m^2 & 0 \\ 0 & -m & 0 \end{bmatrix} \tag{38}$$

$$B^0_{33} = -D^4, \qquad B^2_{33} = 2(m^2-1)D^2, \qquad B^4_{33} = -(m^2-1)^2, \tag{39}$$

with the remaining elements of B^0, B^2 and B^4 all zero.

The exponent q in (31) may now be determined through a consideration of the boundary conditions (21). Introducing the boundary layer coordinates (30) and noting (33), we have

$$u_M(0) + \beta^{q+1}[f(0) + f(l/\beta)] = 0 \tag{40a}$$

$$v_M(0) + \beta^{q+2}[g(0) + g(l/\beta)] = 0 \tag{40b}$$

$$w_M(0) + \beta^q[w_B(l/\beta)] = 0 \tag{40c}$$

$$Dw_M(0) + \beta^{q-1}[Dw_B(0) + Dw_B(l/\beta)] = 0 \tag{40d}$$

and an analogous set of conditions at $x = l$. Now of the eight solutions of (35), only four are regular; hence $\mathbf{u}_M$ contains four constants of integration. Likewise, of the eight solutions of (36) only two satisfy the decay condition (32); hence $\mathbf{v}_B$ contains two constants of integration. As $\beta \to 0$, (40) must yield a consistent set of conditions for the integration constants. In this limit, the second term in each of the brackets may be ignored because of the decay condition. Thus, by symmetry, we must be left at $x = 0$ with two conditions on $\mathbf{u}_M$ and two on $\mathbf{v}_B$. This is possible only if $q = 0$, in which case we have the *limit boundary conditions*

$$u_M(0) = 0, \qquad v_M(0) = 0, \qquad w_M(0) + w_B(0) = 0, \qquad Dw_B(0) = 0. \tag{41}$$

The first two of (41) imply that as the shell grows thinner, its low natural frequencies approach those of a clamped membrane. This, of course, accords with physical intuition. In fact at this preliminary stage we can say even more. Using the results of this section, we can conclude that the membrane and reduced bending energy densities Φ_M and Φ_R are

0(1). But the bending solutions, which dominate the contributions to Φ_R, decay exponentially over a (dimensionless) distance of $0(\beta)$. Reference to Rayleigh's principle (19) then shows that

$$\lambda_n = \Lambda_n + 0(\beta). \tag{42}$$

The aim of the two sections to follow is to give a concrete, quantitative form to (42).

ASYMPTOTIC EXPANSIONS FOR $m = 0(1)$

The qualitative results just discussed suggest we assume representations for λ, $\mathbf{u}_M$ and $\mathbf{v}_B$ of the form

$$\lambda = \Lambda + \beta\lambda^1 + \ldots \tag{43}$$

$$\mathbf{u}_M = \mathbf{u}_M^0 + \beta\mathbf{u}_M^1 + \ldots \tag{44}$$

$$\mathbf{v}_B = \mathbf{v}_B^0 + \beta\mathbf{v}_B^1 + \ldots . \tag{45}$$

Substituting these expressions into (35), (36) and (40), with $q = 0$, and equating to zero coefficients of like powers of β, we obtain the following sequence of differential equations

$$(\eta M + \Lambda I)\mathbf{u}_M^k = -B\mathbf{u}_M^{k-4} - \sum_{j=1}^{k} \lambda^j \mathbf{u}_M^{k-j}, \qquad k = 0, 1, \ldots \tag{46}$$

$$(B^0 + \eta M^0 + \Lambda I)\mathbf{v}_B^k = -(B^2 + \eta M^2)\mathbf{v}_B^{k-2} - B^4\mathbf{v}_B^{k-4} - \sum_{j=1}^{k} \lambda^j \mathbf{v}_B^{k-j} \tag{47}$$

and boundary conditions at $x = 0, l$:

$$u_M^k + f_k = 0 \tag{48a}$$

$$v_M^k + g_{k-2} = 0 \tag{48b}$$

$$w_M^k + w_B^k = 0 \tag{48c}$$

$$Dw_M^{k-1} + Dw_B^k = 0. \tag{48d}$$

In these equations, all terms with a negative index are to be set to zero. In the boundary conditions (48), bending solutions from the opposite end are transcendentally small and hence, asymptotically, make no contribution. (This is not true if $\lambda > 1$.)

DETERMINATION OF Λ_n AND λ_n^1

An algorithm* for computing all the λ^i may be inferred from the work to follow. However, because of the inherent errors in classical shell theory, only Λ_n and λ_n^1 can be considered to have physical significance, so we shall restrict attention to the determination of these two quantities.

* Suggested by O'Malley's treatment of the eigenvalue problem for the stiff string [14].

For what follows, we shall need to fix the magnitude of the eigenvectors. It proves convenient to require

$$\int_0^l \mathbf{u}_M \cdot \mathbf{u}_M \, dx = 1, \tag{49}$$

i.e. to normalize the regular part of the eigenvectors. When (44) is substituted into (49), the resulting expression must be an identity in β. This yields the conditions

$$\int_0^l \mathbf{u}_M^0 \cdot \mathbf{u}_M^0 \, dx = 1 \tag{50}$$

$$\sum_{j=1}^{k} \int_0^l \mathbf{u}_M^j \cdot \mathbf{u}_M^{k-j} \, dx = 0 \qquad k = 1, 2, \ldots. \tag{51}$$

Furthermore, we shall make use of the "Legendre" identity

$$\int_0^l \eta(\bar{\mathbf{u}} \cdot M\mathbf{u} - \mathbf{u} \cdot M\bar{\mathbf{u}}) \, dx = \eta\{[Du + \nu(mv + w)]\bar{u} + \tfrac{1}{2}(1-\nu)(Dv - mu)\bar{v}$$
$$- [D\bar{u} + \nu(m\bar{v} + \bar{w})]u + \tfrac{1}{2}(1-\nu)(D\bar{v} - m\bar{u})v\}_0^l \tag{52}$$

which follows from (23) and holds for any two sufficiently smooth vector fields $\mathbf{u}$ and $\bar{\mathbf{u}}$.

Calculation of Λ_n

For $k = 0$, (46), (48a) and (48b) read

$$[\eta M(D, m, \nu) + \Lambda I]\mathbf{u}_M^0 = 0 \tag{53}$$

$$u_M^0 = v_M^0 = 0, \qquad x = 0, l. \tag{54}$$

The form of the solution for the modes of this eigenvalue problem depends on the nature of the roots of the polynomial

$$-2\det[\eta M(p, m, \nu) + \Lambda I] = (1-\nu)\eta^2(1-\Lambda)p^4 + \eta\Lambda[(1-\nu)\eta(2m^2+3+2\nu) - (3-\nu)\Lambda]p^2$$
$$+ \Lambda[(m^2+1)\eta - \Lambda][2\Lambda - (1-\nu)\eta m^2] \equiv ap^4 + bp^2 + c. \tag{55}$$

If $\Lambda < 1$, $a > 0$, $b > 0$ and $\operatorname{sgn} c = \operatorname{sgn}[2\Lambda - (1-\nu)\eta m^2]$. Thus for $m = 0$, (55) has four imaginary roots, namely

$$\pm i\sqrt{\left(\frac{\Lambda(\eta-\Lambda)}{\eta(1-\Lambda)}\right)}, \qquad \pm i\sqrt{[2(1+\nu)\Lambda]}. \tag{56}$$

For $m = 1$ there are, likewise, four imaginary roots of the form $\pm is_n$, $\pm i\hat{s}_n$, $(s_n, \hat{s}_n) > 0$, so long as $2(1+\nu)\Lambda > 1$; if $2(1+\nu)\Lambda = 1$ there are two imaginary roots and a repeated zero root. If $2(1+\nu)\Lambda < 1$ and $m = 1$ or if $m \geq 2$, there are two imaginary and two real roots of the form $\pm is_n$, $\pm \tilde{s}_n$, $(s_n, \tilde{s}_n) > 0$.

For $m = 0$ there is uncoupling into the longitudinal modes

$$u_M^0 = A_n \sin(s_n x), \qquad w_M^0 = \frac{-\nu s_n A_n}{1-\mu_n}\cos(s_n x), \tag{57}$$

where

$$\mu = (1-\nu^2)\Lambda = \Lambda/\eta, \tag{58}$$

$$\mu_n = \tfrac{1}{2}\{1+s_n^2-\sqrt{[(1-s_n^2)+4\nu^2 s_n^2]}\}, \tag{59}$$

$$A_n = \left\{\frac{l}{2}\left[1+\left(\frac{\nu s_n}{1-\mu_n}\right)^2\right]\right\}^{-\frac{1}{2}} \tag{60}$$

and $s_n = n\pi/l$, and the torsional modes

$$v_M^0 = \sqrt{(2/l)}\sin(s_n x), \tag{61}$$

where

$$\Lambda_n = \frac{s_n^2}{2(1+\nu)}. \tag{62}$$

For $m \geq 1$, it is expedient, for computational purposes, to separate the natural modes into those with v and w symmetric or antisymmetric with respect to the center of the shell. From (23) and the preceding discussion of the roots of (55), it follows that if $2(1+\nu)\Lambda > 1$ and $m = 1$, the symmetric modes, in an obvious notation, are of the form

$$u_M^0 = s_n[m^2-\nu s_n^2+2\nu(1+\nu)\Lambda_n]A_n \sin s_n(x-l/2)+(s_n, A_n \leftarrow \hat{s}_n, \hat{A}_n). \tag{63}$$

$$v_M^0 = -m[m^2+(2+\nu)s_n^2-2(1+\nu)\Lambda_n]A_n \cos s_n(x-l/2)+(s_n, A_n \leftarrow \hat{s}_n, \hat{A}_n). \tag{64}$$

$$\begin{aligned}[1-(1-\nu^2)\Lambda_n]w_M^0 = {} & [m^4+2m^2s_n^2+\nu^2s_n^4-2(1+\nu)(m^2+\nu^2s_n^2)\Lambda_n]A_n \cos s_n(x-l/2) \\ & +(s_n, A_n \leftarrow \hat{s}_n, \hat{A}_n),\end{aligned} \tag{65}$$

while if $2(1+\nu)\Lambda < 1$ and $m = 1$ or if $m \geq 2$, we have identical expressions except that $\hat{s}_n$ is everywhere replaced by $i\tilde{s}_n$ and $\hat{A}_n$ is everywhere replaced by $\tilde{A}_n$. An analogous set of formulae with the sines and cosines reversed and u_M^0 replaced by $-u_M^0$ holds for the antisymmetric modes. The modes for the special case $2(1+\nu)\Lambda = 1$, $m = 1$ may be easily written down, but shall not be listed here.

A 2×2 transcendental determinantal equation for Λ_n is obtained by substituting (63) and (64) (or the corresponding expressions for $2(1+\nu)\Lambda < 1$ and $m = 1$ or $m \geq 2$) into the boundary conditions (54) and requiring A_n and $\hat{A}_n$ (or A_n and $\tilde{A}_n$) to be non-zero. An analogous determinantal equation is obtained for the values of Λ_n associated with the antisymmetric modes. The eigenvalues $\Lambda_1, \ldots, \Lambda_6$ have been computed for $m = 1, 2, 3, 4,$ 5 and 6, $l = 1, 1{\cdot}5, 2, 3, 4$ and 5, and $\nu = 0{\cdot}3$ and are listed in the table. As a rule, the eigenvalues associated with even and odd modes interlace, although the column for $m = 1$, $l = 5$ in the table shows that there are exceptions.

Calculation of λ_n^1

To compute λ^1, we must consider the solutions for $\mathbf{u}_M^1$ and $\mathbf{v}_B^0$. From (46) to (48) we have

$$(\eta M+\Lambda I)\mathbf{u}_M^1 = -\lambda^1\mathbf{u}_M^0 \tag{66}$$

$$(B^0+\eta M^0+\Lambda I)\mathbf{v}_B^0 = 0, \tag{67}$$

and, at $x = 0, l$,

$$u_M^1 + f_0 = 0, \qquad v_M^1 = 0 \tag{68a, b}$$

$$w_M^0 + w_B^0 = 0, \qquad Dw_B^0 = 0. \tag{69a, b}$$

An expression for λ^1 may be obtained without explicitly solving for $\mathbf{u}_M^1$ as follows:

First, take the dot product of $\mathbf{u}_M^1$ with (53), subtract the dot product of $\mathbf{u}_M^0$ with (66) and integrate from $x = 0$ to $x = l$ to obtain

$$\int_0^l [\mathbf{u}_M^1 \cdot (\eta M + \Lambda I)\mathbf{u}_M^0 - \mathbf{u}_M^0 \cdot (\eta M + \Lambda I)\mathbf{u}_M^1]\, \mathrm{d}x = \lambda^1 \int_0^l \mathbf{u}_M^0 \cdot \mathbf{u}_M^0\, \mathrm{d}x. \tag{70}$$

Next, apply (50), (52) and the boundary conditions (54) and (68) to reduce this relation to

$$\begin{aligned} \lambda^1 &= -\eta\{(Du_M^0 + \nu w_M^0) f_0\}_0^l \\ &= 2\eta[Du_M^0(0) + \nu w_M^0(0)] f_0(0), \end{aligned} \tag{71}$$

where the last line is a consequence of the modes being symmetric or antisymmetric about $x = l/2$.

The only unknown in (71) is $f_0(0)$. To express this value in terms of $\mathbf{u}_M^0$, we solve (67) subject to the decay condition (32) and the boundary conditions (69). With M^0 and B^0 given by (37) and (39), we find that

$$f_0(y) = (\nu/\kappa) w_M^0(0)\, \mathrm{e}^{-\kappa y} \cos \kappa y, \tag{72}$$

where

$$\kappa = \left(\frac{1-\Lambda}{4}\right)^{\frac{1}{4}}. \tag{73}$$

Substituting (72) into (71), with η and κ given by (11) and (73), we obtain an expression for λ^1 in terms of the solution of the first approximation membrane problem alone:

$$\lambda_n^1 = \frac{2\sqrt{(2)}\nu w_M^0(0)}{(1-\Lambda_n)^{\frac{1}{4}}}[Du_M^0(0) + \nu w_M^0(0)]. \tag{74}$$

For given values of l and m, the numerical value of λ_n^1 can be computed by obtaining a value of Λ_n from the table, determining the associated value of s_n from (55), the ratio $A_n/\hat{A}_n$ from either of the boundary conditions (54), the value of A_n from the normalization condition (50) and finally, the values of Du_M^0 and w_M^0 from (63) and (65) or their counterparts. As expected, (74) breaks down as the transition value $\Lambda = 1$ is approached.

ASYMPTOTIC FORMS FOR LARGE m

Figure 5 of [10] shows, for a shell of given length l, that the smallest eigenvalue, call it λ_*, occurs at a circumferential wave number m_* which grows as $\beta \to 0$. Moreover, for fixed β, m_* is seen to decrease with increasing l. These numerical results may be further quantified by a simple, analytical upper bound calculation which suggests

$$\lambda_* = 0(\beta^2/l^2), \qquad m_* = 0(1/\beta l), \qquad p_R = 0(1/l), \tag{75}$$

if l is not too small compared to unity. Here p_R denotes one of the regular roots of (25), i.e. one of the roots that remains bounded as $\beta \to 0$.

If we wish to compute λ_*, an asymptotic expansion differing from the one of the preceding sections must be employed. In this section we sketch the steps leading to a first-approximation expression for λ_* and the other nearby eigenvalues. For brevity, we do not attempt to set up a formal expansion procedure. The final result is a remarkably simple formula involving the eigenvalues of a clamped–clamped beam and having a region of overlap with the membrane approximation.

Assuming (75) to be valid—as the consistency of the subsequent analysis shows it to be—we find, by inspection, that (25) has four regular roots of the form

$$p_R = l^{-1}[p_R^0 + 0(\beta/l)], \tag{76}$$

where

$$(p_R^0)^4 + \beta^4 m^4 (m^2-1)^2 - \lambda m^2(m^2+1) = 0, \tag{77}$$

and four singular roots of the form

$$p_S = \beta^{-1}[p_S^0 + 0(\beta/l)], \tag{78}$$

where

$$(p_S^0)^4 + 1 = 0. \tag{79}$$

It is convenient again to introduce the boundary layer coordinates (30) and, in addition, the interior coordinate

$$\xi = x/l. \tag{80}$$

The solution of (22) may be then represented in the form

$$\mathbf{u} = \mathbf{u}_R(\xi) + \mathbf{u}_S(y) + \mathbf{u}_S(z), \tag{81}$$

where the singular solutions must approach zero with increasing argument. An order of magnitude analysis shows that $|\mathbf{u}_R|$, $|\mathbf{u}_S| = 0(1)$ and that the components of these vectors satisfy boundary conditions at $x = 0, l$ of the form

$$u_R + 0(\beta/l) = 0, \qquad v_R + 0(\beta/l)^2 = 0 \tag{82a, b}$$

$$w_R + w_S + 0(\beta/l) = 0, \qquad Dw_S + 0(\beta/l) = 0. \tag{82c, d}$$

As before, the boundary conditions show that the singular (or boundary-layer) solutions have no effect to a first approximation.

Let

$$(\alpha_n/l)^4 \equiv \lambda_n m^2(m^2+1) - \beta^4 m^4(m^2-1)^2. \tag{83}$$

Then, in view of (75) and (77), the regular solutions of (22) symmetric about the center of the shell have the form

$$u_R = -\alpha_n[B_n \sin \alpha_n(\xi - \tfrac{1}{2}) + C_n \sinh \alpha_n(\xi - \tfrac{1}{2})] + 0(\beta/l) \tag{84}$$

$$v_R = m[B_n \cos \alpha_n(\xi - \tfrac{1}{2}) + C_n \cosh \alpha_n(\xi - \tfrac{1}{2})] + 0(\beta/l) \tag{85}$$

$$w_R = -m^2[B_n \cos \alpha_n(\xi - \tfrac{1}{2}) + C_n \cosh \alpha_n(\xi - \tfrac{1}{2})] + 0(\beta/l). \tag{86}$$

When (84) and (85) are substituted into the boundary conditions (82a, b), the resulting 2×2 frequency determinant is identical to that for a clamped–clamped beam executing symmetric vibrations.

An analogous statement holds for the antisymmetric modes. In both cases we obtain the well-known condition [15]

$$\cos\alpha_n \cosh\alpha_n = 1\begin{cases} n = 1, 3, \ldots, \text{symmetric modes} \\ n = 2, 4, \ldots, \text{antisymmetric modes}. \end{cases} \tag{87}$$

The first four solutions of (87) are [15]: $\alpha_1 = 4{\cdot}7300, \alpha_2 = 7{\cdot}8532, \alpha_3 = 10{\cdot}9956, \alpha_4 = 14{\cdot}1372$. From (83), the desired asymptotic formula for the eigenvalues is therefore

$$\lambda_n = \frac{(\alpha_n/l)^4}{m^2(m^2+1)} + \frac{\beta^4 m^2(m^2-1)^2}{m^2+1} + O(\beta/l). \tag{88}$$

Yu [16] and Weingarten [17] have obtained a similar result, but without the error term. Both authors start with Donnell-type equations and arrive at their approximate frequency equations via certain *ad-hoc* assumptions.

The circumferential wave number m_* which minimizes the right hand side of (88) clearly has the order of magnitude relation to β and l assumed in (75). If $\beta l \ll 1$, then

$$m_*^2 \simeq \frac{\alpha_1}{\beta l}, \qquad \lambda_* \simeq \frac{2\alpha_1^2\beta^2}{l^2}. \tag{89}$$

The graph of (88) for the appropriate values of m_* is virtually indistinguishable from the one presented by Forsberg [10].

If we retain the terms in common between (55) and (77) and the associated expressions for the components of $\mathbf{u}_M^0$ and $\mathbf{u}_R$ and estimate the resulting errors, we are left with, simply

$$\lambda_n = \frac{(\alpha_n/l)^4}{m^2(m^2+1)}\left[1 + O\left(\frac{1}{m^2 l^2}, \beta^4 l^4 m^{10}\right)\right]. \tag{90}$$

Thus the membrane approximation Λ_n and (88) agree for shells of sufficient length and thinness, although, of course, to compute the minimum natural frequency for a shell of given length, (88) must be used.

CONCLUSIONS

To complete the picture of the frequency spectrum, it is necessary to investigate the solutions of the governing equations (21) and (22) near and above the transition value $\lambda = 1$. The analysis near $\lambda = 1$ is complicated by the fact that solving for the rapidly varying portion of $\mathbf{u}$ leads to the consideration of a bi-cubic equation rather than a simple bi-quadratic. For values of λ somewhat above the transition value, solving for the rapidly varying part of $\mathbf{u}$ again leads to the consideration of a simple bi-quadratic equation, but now with some purely imaginary roots, implying that rapid axial variations extend over the entire shell. The influence of the bending stiffness on the basic membrane frequencies does not seem so clear here as with the low natural frequencies. The matter awaits further study.

The analysis presented herein could be extended to rectangular cylindrical panels. However, for low frequencies, the boundary layers along the straight edges of the panels (the cylinder generators) will be wider, of order $(R^3h/l^2)^{\frac{1}{4}}$, as compared to the boundary layers of order $(hR)^{\frac{1}{2}}$ at the ends of the panels. In addition, complicated corner layers appear, but their effects on the low frequencies can be expected to be negligible.

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Абстракт—Применяя асимптотический метод, вычисляются низкие, собственные частоты для полно защемленной, упруго изотропной, круглой, цилиндрической оболочки. Исследуются два типа разложений. Первое применяет форму заданного, низкого числа кольцевых волн. Для первого приближения получаются частоты на основе мембранной теории. Приводится таблица численных значений для разных длин оболочки, числа кольцевых волн и значения коэффициента Пуассона 0, 3. Поправка вследствие изгиба является пропорциональной к квадратическому корню отношения толщины к радиусу, стремится к нулю с коэффициентом Пуассона и яьляется выразимой исключлтельно членами решений мембранной задачи на собственные значения. Второй тип разложения является необходимым для расчета минимальной собственной частоты для оболочки заданной длины и допуекает числу кольцевых волн стремиться к бесконечности, когда отношение толщина—радиус приближается к нулю. Получается замечательно простая формула для частот, выраженных через собственные частоты полно замещенной балки. Указано, что два асимптотические результаты частично совпадают.